Journal of Econometrics and Statistics
Vol. 5, Issue 2, 2025, pp. 163-183

Statistical Properties of Two Asymmetric Stochastic Volatility in Power Mean Models

Antonis Demos

Department of International European Economic Studies, AUEB, Athens,Greecel

ABSTRACT
Here we investigate the statistical properties of two autoregressive normal asymmetric SV models with possibly time varying means. These, although they seem very similar, it turns out, that they possess quite different statistical properties. The derived properties can be employed to develop tests or to check for up to forth order stationarity, something important for the asymptotic properties of various estimators.

KEYWORDS
Gaussian Stochastic Volatility, in Mean, static and dynamic statistical properties, financial returns..

1. Introduction

Empirical research in economics and finance has identified several persistent statistical patterns in asset return dynamics, often labeled as stylized facts. Chief among these is volatility clustering, wherein episodes of heightened (or diminished) volatility are followed by similar periods. Originally documented in financial time series, this phenomenon has analogs in the physical sciences, such as turbulence—referred to there as indeterminacy (Barndorff-Nielsen [3]).

The recognition of volatility clustering has significantly influenced time series modeling. While early responses to this empirical regularity centered around ARCH (Autoregressive Conditional Heteroskedasticity) and GARCH-type (Generalized ARCH) (e.g., Engle [12] ; Bollerslev [7]), recent developments have increasingly turned toward Stochastic Volatility (SV) models (Taylor [38]). In contrast to GARCH models, which rely on deterministic volatility updating rules driven by past shocks, SV models feature a separate stochastic process governing volatility, with SV-M (SV in Mean) models introducing a dynamic link between expected returns and conditional volatility.

These models are particularly well suited to capturing key economic regularities. In equity markets, for instance, theory frequently posits a risk-return trade-off, whereby expected returns depend on conditional variance (e.g., Merton [29]; Glosten, Jagannathan, and Runkle [16] GJR hereafter). SV-M models provide a natural structure for modeling such relationships by allowing conditional variance (or standard devia-

CONTACT. Email: demos@aueb.gr

Article History
Received : 18 January 2025; Revised : 19 February 2025; Accepted : 22 February 2025; Published : 28 June 2025

To cite this paper

Antonis Demos (2025). Statistical Properties of Two Asymmetric Stochastic Volatility in Power Mean Models. *Journal of Econometrics and Statistics.* 5(2), 163-183.

tion) to enter the mean equation, thereby capturing time-varying risk premia. Unlike GARCH-M frameworks, SV-M specifications can accommodate ex-ante relationships and complex feedback mechanisms between volatility and returns (Koopman and Uspensky [26]).

Volatility feedback, as articulated in Campbell and Hentschel [8], suggests that unexpected shocks raise expected future volatility, which—if positively priced—leads to immediate price drops. SV-M models capture this mechanism more fundamentally than GARCH variants by embedding correlation between return and volatility innovations, a feature not intrinsic to the standard GARCH framework.

Another prominent empirical feature in financial data is asymmetry in volatility response—notably, volatility increases more after negative shocks than after equally sized positive ones (Black [5]). While this leverage effect has traditionally been modeled using asymmetric GARCH variants (e.g., Exponential GARCH of Nelson [31], GJR-GARCH or Quadratic GARCH of Sentana [37]), asymmetric SV models offer a more structurally coherent alternative. In the SV context, asymmetry arises through a negative correlation between return and volatility shocks, as emphasized by Jacquier, Polson, and Rossi [24], enabling a more direct and interpretable modeling of asymmetry.

Despite their theoretical appeal, SV models were historically underutilized due to challenges in estimation—especially since volatility is not directly observed. However, this has changed with advances in estimation techniques. Bayesian methods (Jacquier, Polson, and Rossi [23], [24]; Kim, Shephard, and Chib [25]), simulation-based inference (Gallant and Tauchen [14]; Gourieroux, Monfort, and Renault [17]), and approximate likelihood approaches (e.g., data cloning by Bermudez, Marin, and Veiga [4]; Laplace approximations by Marin, Romero, and Veiga [28]) have enabled more robust inference within SV frameworks.

This study contributes to this growing body of research by examining the statistical properties of asymmetric SV-PM models, where the conditional variance, raised to a power, directly affects the return dynamics, and the return and volatility innovations are potentially correlated. We consider two first-order autoregressive SV structures and highlight their distinct statistical behavior, offering insights into their appropriateness for modeling financial return series with dynamic risk premia.

Foundational treatments of SV models can be found in Taylor [38], Harvey, Ruiz, and Shephard [21], and Jacquier, Polson, and Rossi [23]. For extended treatments of asymmetry, see Tsiotas [39]; for long-memory SV models, see Ghysels, Harvey, and Renault [15] and Harvey [19]. The Generalized Asymmetric SV-M model by Mao et al. [27], which introduces asymmetry via mean innovation transformation, represents a further innovation, though in this study we adhere to the classical approach wherein asymmetry is introduced through innovation correlation, as in Jacquier, Polson and Ross [24] (JPR here after).

To our knowledge, this is the first formal derivation of the statistical properties of the asymmetric SV-PM model with normally distributed standardized innovations and correlation-driven asymmetry. In doing so, we aim to offer a more flexible and theoretically consistent alternative to GARCH-based approaches for modeling volatility dynamics and the risk-return relationship in financial markets.

In the next section we present the two SV-PM models and derive their static and dynamic moments. In the final section we compare the properties of the two models and conclude.

2. The Two SV-PM Models

We consider the following normal Autoregressive Stochastic Volatility in Mean class of models:

$$y_t = c + \lambda\sigma_t^{2\alpha} + \varepsilon_t^* = c + \lambda\sigma_t^{2\alpha} + \varepsilon_t\sigma_t \quad where, \tag{1}$$

$$\ln\sigma_t^2 = \omega + \psi\ln\sigma_{t-1}^2 + \eta_{t-1}\ (SV1)\ and \tag{2}$$

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho\sigma_\eta \\ \rho\sigma_\eta & \sigma_\eta^2 \end{pmatrix}\right).$$

We call this model SV1-PM. Notice that the findings of Carnero, Pena and Ruiz [9] suggest that the assumption of normality is reasonably appropriate for financial time series. The estimation of the model above, with $\alpha = 0.5$, can be found in Arvanitis and Demos [1], whereas restricted version of it, with $c = \lambda = 0$, has been estimated by quasi maximum likelihood in Harvey and Shephard [22], and by MCMC in Meyer and Yu [30]. Further, Asai and McAleer [2] present some properties of the restricted model concerning mainly the asymmetry and the leverage effect.

However, a second model has been considered in applied work. Specifically, instead of the above conditional variance specification 2 the following one is employed :

$$\ln\sigma_t^2 = \omega + \psi\ln\sigma_{t-1}^2 + \eta_t\ \text{(SV2)}. \tag{3}$$

We name this one the SV2-PM model. A similar model, with $\rho = 0$ and $\alpha = 0.5$, has been estimated in Koopman and Uspensky [26] by simulated maximum likelihood, but they add an autoregressive term in the mean. Further, a model with $c = \lambda = 0$ and $\alpha = 1$, but with non-normal error distribution, has been employed by JPR, whereas under the same restrictions but with an autoregressive term Romero and Ropero-Moriones [34] employ data cloning estimation method.

Although the two models look very similar, there are important differences between the statistical properties that they can accommodate. In fact, notice that for the SV1-M model the mean error and the conditional variance are contemporaneously uncorrelated, which is not the case for the SV2-PM one. Nevertheless, Yu [40] proved that the partial derivative of future volatility with respect to the error is not necessarily negative when $\rho < 0$, i.e. it could be the case that even if $\rho < 0$ future volatility could decrease with a negative error, claiming the the variance specification in (2) is a more "natural" one (see details in Yu [40]).

Now, form equation (1), and for $\alpha = 1$, we get

$$y_t = c + \lambda E_{t-1}\left(\sigma_t^2\right) + \lambda\left(\sigma_t^2 - E_{t-1}\left(\sigma_t^2\right)\right) + \varepsilon_t\sigma_t \quad where,$$

$E_{t-1}\left(\sigma_t^2\right)$ is the expected volatility given information at time $t-1$. Hence λ could also represent the volatility feedback coefficient, as the term $\sigma_t^2 - E_{t-1}\left(\sigma_t^2\right)$ is the unexpected part for volatility. In fact, this is a restricted version of the model considered in Campbell and Hentschel [8], where the risk premium and the feedback coefficients are different. In a GARCH-M type model, this parameterization is not possible as $E_{t-1}\left(\sigma_t^2\right) = \sigma_t^2$, and consequently this constitutes a comparative advantage of the SV-PM model (see Koopman and Uspensky [26] for more details).

Let us now explore the properties of these models. Notice that, to express the static and dynamic moments of the models, as a function of the parameters, we expand y_t

employing equation (1) as well as equation (2) or equation (3) and use the formulae in Appendix A and Appendix B (please see Demos [11] for analytic proofs).

2.1. *Properties of the SV1-PM Model*

Let us now investigate the statistical static and dynamic properties of the $SV1-PM$ model.

2.1.1. Static Propertiesl

First, it is easy to prove that skewness of $\varepsilon_t^* = \varepsilon_t\sigma_t$, $sk\left(\varepsilon_t^*\right)$, is 0 and the kurtosis coefficient is given by

$$\kappa_{\varepsilon_t^*} = \frac{3E\left(\sigma_t^4\right)}{\left(V\left(\varepsilon_t\sigma_t\right)\right)^2} = 3\exp\left(\frac{\sigma_\eta^2}{1-\psi^2}\right) > 3,$$

which is bigger than 3, i.e. the stochastic volatility increases the kurtosis of the errors, a well known fact of the SV models.

Now

$$E\left(y_t\right) = c + \lambda\exp\left[\frac{\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]. \tag{4}$$

Notice that for $c = 0$ the unconditional risk has the sign of λ, positive, as λ represents the price of risk, in Financial Economics, where as for $c = \lambda = 0$, as in Harvey and Shephard [22], the risk premium is zero.

Further,

$$\begin{aligned} V\left(y_t\right) = \lambda^2 &\left\{\exp\left[\frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right] - 1\right\}\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right] \\ &+ \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]. \end{aligned} \tag{5}$$

It is worth mentioning that the price of risk parameter λ increases the variance of the observed process, independent of the sign of the price of risk parameter λ.

The skewness coefficient of observed process y_t is given by

$$\begin{aligned} sk\left(y_t\right) = {}& \lambda\frac{\lambda^2\left\{\exp\left[\frac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 3\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] + 2\right\}\exp\left[\frac{3\alpha\omega}{(1-\psi)} + \frac{3\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right]}{\left[V\left(y_t\right)\right]^{3/2}} \\ &+3\lambda\frac{\left\{\exp\left[\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{(\alpha+1)\omega}{(1-\psi)} + \frac{\left(\alpha^2+1\right)\sigma_\eta^2}{2(1-\psi^2)}\right]}{\left[V\left(y_t\right)\right]^{3/2}}, \end{aligned} \tag{6}$$

where $V\left(y_t\right)$ is given in (5). Notice that the skewness coefficient has the sign of λ, i.e. positive under the assumption of risk premium positivity, as it is highly unlikely the first part of the expression to be negative and bigger in absolute value to the second one. Further, for $\lambda = 0$ the skewness is zero.

Now the kurtosis coefficient of y_t equals:

$$\kappa(y_t) = \frac{\lambda^4\left\{\exp\left[\frac{6\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 4\exp\left[\frac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] + 6\exp\left[\frac{2\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right] - 3\right\}\exp\left[\frac{4\alpha\omega}{(1-\psi)} + \frac{2\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]}{\left[V(y_t)\right]^2}$$

$$+\frac{6\lambda^2\left\{\left\{\exp\left[\frac{\alpha(\alpha+1)\sigma_\eta^2}{(1-\psi^2)}\right] - 2\right\}\exp\left[\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right] + 1\right\}\exp\left[\frac{(2\alpha+1)\omega}{(1-\psi)} + \frac{\left(2\alpha^2+1\right)\sigma_\eta^2}{2(1-\psi^2)}\right]}{\left[V(y_t)\right]^2}$$

$$+3\frac{\exp\left[\frac{2\omega}{(1-\psi)} + \frac{4\sigma_\eta^2}{2(1-\psi^2)}\right]}{\left[V(y_t)\right]^2}, \tag{7}$$

where $V(y_t)$ is, again, given in (5).

Further,

$$V\left(y_t^2\right) = \mathbf{A}\lambda^3\exp\left[\frac{3\alpha\omega}{(1-\psi)} + \frac{5\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right] + 2\mathbf{B}\lambda^2\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \tag{8}$$

$$+4c\lambda\left\{3\exp\left[\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{(\alpha+1)\omega}{(1-\psi)} + \frac{\left(\alpha^2+1\right)\sigma_\eta^2}{2(1-\psi^2)}\right]$$

$$+4c^2\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right] + \left\{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{(1-\psi^2)}\right],$$

where

$$\mathbf{A} = \lambda\left\{\exp\left[\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{\alpha\omega}{(1-\psi)} + \frac{3\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right] + 4c\left\{\exp\left[\frac{2\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \textit{ and}$$

$$\mathbf{B} = 2c^2\left\{\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} + \left\{3\exp\left[\frac{2\alpha\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{\omega}{(1-\psi)} + \frac{\left(2\alpha^2+1\right)\sigma_\eta^2}{2(1-\psi^2)}\right].$$

2.1.2. Dynamic Properties

For the autocovariances of the observed process, y_t, we have that:

$$Cov(y_t, y_{t-k}) = \lambda^2\left[\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right] - 1\right]\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]$$

$$+\lambda\alpha\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(2\alpha+1)\omega}{2(1-\psi)} + \frac{\sigma_\eta^2\left(4\alpha^2+4\alpha\psi^k+1\right)}{8(1-\psi^2)}\right]$$

and the correlations are given by dividing the above expression by the variance in (5). It is worth noticing that, provided that λ is positive then the correlations can be either positive or negative, depending on the relative values of of the parameters. Notice that for $\rho = 0$ the $Corr(y_t, y_{t-k})$ is proportional to λ^2, whereas for $\lambda = 0$ the autocorrelations are zero.

In terms of leverage effect we have that

$$Corr\left(\sigma_t^2, \varepsilon_{t-k}^*\right) = \rho\sigma_\eta\psi^{k-1}\frac{\exp\left[\frac{4\psi^k-1}{8(1-\psi^2)}\sigma_\eta^2\right]}{\sqrt{\left(\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)}} \tag{9}$$

and has the sign of ρ, i.e. the leverage effect can be satisfied by the model if and only if $\rho < 0$, provided that $\psi > 0$ something which is very plausible due to volatility clustering (see below).

In terms of dynamic asymmetry, as we call $Cov\left(y_t^2, y_{t-k}\right)$, first expand y_t^2 and y_{t-k}, employing equations 1 and 2 (see Demos [11]) and using equations 14 and 15, we get:

$$Cov\left(y_t^2, y_{t-k}\right) = \mathbf{A}\lambda^2\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]$$

$$+\mathbf{B}\lambda\exp\left[\frac{(2\alpha+1)\,\omega}{2\,(1-\psi)} + \frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{8\,(1-\psi^2)}\right] + \rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3\omega}{2\,(1-\psi)} + \frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8\,(1-\psi^2)}\right].$$

where

$$\mathbf{A} = \lambda\left(\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)\exp\left[\alpha\omega + \frac{3\alpha^2\sigma_\eta^2}{2\,(1-\psi^2)}\right] + 2c\left(\exp\left[\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)$$

$$\mathbf{B} = 2\rho\sigma_\eta\psi^{k-1}\left\{\lambda\alpha\exp\left[\frac{\alpha\omega}{1-\psi} + \frac{\alpha\left(\psi^k+3\alpha\right)\sigma_\eta^2}{2\,(1-\psi^2)}\right] + c\right\}\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{2\,(1-\psi^2)}\right)$$

$$+\left[\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{1-\psi^2}\right)-1\right]\exp\left[\frac{\omega}{2\,(1-\psi)} + \frac{3\sigma_\eta^2}{8\,(1-\psi^2)}\right]$$

Now for $\lambda = 0$ we get that $Corr\left(y_t^2, y_{t-k}\right)$ is proportional to $\rho\sigma_\eta\psi^{k-1}$ and consequently has the sign of ρ, and, of course, it is independent of α. Notice that in this case and for $\rho < 0$ we have that $Corr\left(y_t^2, y_{t-k}\right) > Corr\left(\sigma_t^2, \varepsilon_{t-k}^*\right)$, i.e. the leverage effect is stronger than the dynamic asymmetry. Finally, for $c = 0$, it possible to have $Cov\left(y_t^2, y_{t-k}\right) > 0$ for $\rho < 0$, i.e. to have the leverage effect, but positive dynamic asymmetry.

Employing equation 14, in Appendix A, the volatility clustering is given by

$$Corr\left(\sigma_t^2, \sigma_{t-k}^2\right) = \frac{\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1}{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1}, \tag{10}$$

and it is the same with the one of the SV2-PM specification.

Further,

$$Corr\left(\varepsilon_t^{*2}, \varepsilon_{t-k}^{*2}\right) = \frac{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1}{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1}. \tag{11}$$

For the dynamic kurtosis, as we call here $Cov\left(y_t^2, y_{t-k}^2\right)$, employ equations 1 for expand y_t^2 and y_{t-k}^2 (see the Supplement) and employ equations 14 and 15 of Appendix A to get:

$$
\begin{aligned}
Cov\left(y_t^2, y_{t-k}^2\right) = \lambda^4 &\left\{\exp\left[\frac{4\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{4\alpha\omega}{(1-\psi)}+\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \\
+2\lambda^2&\left\{\left(1+2\alpha^2\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{2\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{(2\alpha+1)\,\omega}{(1-\psi)}+\frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{2\,(1-\psi^2)}\right] \\
&+4\mathbf{A}c\lambda^2\exp\left[\frac{2\alpha\omega}{(1-\psi)}+\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]+2\mathbf{B}\lambda\exp\left[\frac{(2\alpha+1)\,\omega}{2\,(1-\psi)}+\frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{8\,(1-\psi^2)}\right] \\
+4\mathbf{D}c\lambda\alpha\rho\sigma_\eta\psi^{k-1}&\exp\left[\frac{(2\alpha+1)\,\omega}{2\,(1-\psi)}+\frac{\sigma_\eta^2\left(4\alpha\psi^k+1+4\alpha^2\right)}{8\,(1-\psi^2)}\right]+\mathbf{F}\exp\left[\frac{3\omega}{2\,(1-\psi)}+\frac{5\sigma_\eta^2}{8\,(1-\psi^2)}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{A} &= \lambda\left\{\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{\alpha\omega}{(1-\psi)}+\frac{3\alpha^2\sigma_\eta^2}{2\,(1-\psi^2)}\right]+c\left\{\exp\left[\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}, \\
\mathbf{B} &= 2\lambda^2\alpha\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{2\alpha\omega}{(1-\psi)}+\frac{\alpha\left[2\,(2\alpha+1)\,\psi^k+4\alpha+1\right]\sigma_\eta^2}{2\,(1-\psi^2)}\right] \\
&+c\left\{\left(2+\alpha^2\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-2\right\}\exp\left[\frac{\omega}{2\,(1-\psi)}+\frac{3\sigma_\eta^2}{8\,(1-\psi^2)}\right], \\
\mathbf{D} &= \lambda\left\{\exp\left[\frac{\alpha\left(\psi^k+2\alpha\right)\sigma_\eta^2}{2\,(1-\psi^2)}\right]+\exp\left[\frac{\alpha\left(2\alpha\psi^k+1\right)\sigma_\eta^2}{2\,(1-\psi^2)}\right]\right\}\exp\left[\frac{\alpha\omega}{(1-\psi)}+\frac{\alpha^2\sigma_\eta^2}{2\,(1-\psi^2)}\right]+c,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{F} = &\left\{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{\omega}{2\,(1-\psi)}+\frac{3\sigma_\eta^2}{8\,(1-\psi^2)}\right] \\
&+2\rho\sigma_\eta\psi^{k-1}\left\{\lambda\exp\left[\frac{\alpha\omega}{(1-\psi)}+\frac{\alpha\left(2\psi^k+\alpha+1\right)\sigma_\eta^2}{2\,(1-\psi^2)}\right]+c\right\}\exp\left[\frac{\psi^k\sigma_\eta^2}{2\,(1-\psi^2)}\right].
\end{aligned}
$$

Employing equation 8 for $Var\left(y_t^2\right)$ we can get the autocorrelation function, $Corr\left(y_t^2, y_{t-k}^2\right)$.

Notice that for $\lambda = c = 0$ we get

$$
Corr\left(y_t^2, y_{t-k}^2\right) = \frac{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1}{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1},
$$

which is the same as equation 18 in Perez, Ruiz and Veiga [33] and equation 5 in Ruiz and Veiga [36], and again it is positive for any values of ρ.

2.2. *Properties of the SV2-PM Model*

2.2.1. Static Moments

Now, if instead the conditional variance specification we employ the specification in JPR, equation 3, we get

$$E\left(y_t\right) = c + \lambda \exp\left(\frac{\alpha\omega}{1-\psi} + \frac{\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) + \frac{\rho\sigma_\eta}{2}\exp\left(\frac{\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

and

$$\begin{aligned} Var\left(y_t\right) &= \lambda^2\left[\exp\left(\frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right)-1\right]\exp\left(\frac{2\alpha\omega}{\left(1-\psi\right)} + \frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right)\\ &+\lambda\rho\sigma_\eta\left[\left(2\alpha+1\right)\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right]\exp\left(\frac{\left(2\alpha+1\right)\omega}{2\left(1-\psi\right)} + \frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)\\ &+\left\{\left(1+\rho^2\sigma_\eta^2\right)\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-\frac{\left(\rho\sigma_\eta\right)^2}{4}\right\}\exp\left(\frac{\omega}{\left(1-\psi\right)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right). \end{aligned}$$

Notice that if $\rho = 0$ we get that the expected value and variance of y_t are the same as those for the SV1-PM one (see equations 4 and 5). Further, if $c = \lambda = 0$, as in JPR, $E\left(y_t\right)$ is negative, provided that $\rho < 0$ (leverage effect).

The skewness coefficient of the mean error $\varepsilon_t^* = \sigma_t\varepsilon_t$ is given by

$$sk\left(\varepsilon_t^*\right) = \frac{3}{2}\rho\sigma_\eta\frac{\left\{\left(3+\left(\frac{3}{2}\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)-\left(1+\left(\rho\sigma_\eta\right)^2\right)\right\}\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)+\frac{1}{6}\left(\rho\sigma_\eta\right)^2}{\left(\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right)^{3/2}}. \tag{12}$$

It is worth noticing that although the distribution of the standardized errors is normal the skewness of the mean error is non-zero, and if fact it is negative, provided that $\rho < 0$ (leverage effect).

Now the kurtosis of the mean error is given by

$$\begin{aligned} \kappa\left(\varepsilon_t^*\right) &= \frac{\left[3+24\left(\rho\sigma_\eta\right)^2+16\left(\rho\sigma_\eta\right)^4\right]\exp\left(\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right)}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2}\\ &+3\left(\frac{\rho\sigma_\eta}{2}\right)^2\frac{\left\{2\left(1+\left(\rho\sigma_\eta\right)^2\right)-\left(3+\left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)\right\}\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2}, \end{aligned}$$

and for $\rho = 0$, as in Koopman and Uspensky [26], $\kappa\left(\varepsilon_t^*\right)$ is the same as for the SV1-PM model.

The asymmetry of the observed process, y_t, is very complicated for the full model and it is presented in Appendix B.To get the asymmetry of the observed process, y_t, expand $E\left(y_t - E\left(y_t\right)\right)^3$ and employing the appropriate formulae to substitute $E\left(\sigma_t^s\right)$

and $E\left(\varepsilon_t^k \sigma_t^l\right)$ (see Appendix B) for various values of s, k and l we get (see Demos [11]):

$$E\left(y_t - E\left(y_t\right)\right)^3 = \lambda^3 \left[\exp\left(\frac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) - 3\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) + 2\right] \exp\left(\frac{3\alpha\omega}{1-\psi} + \frac{3\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)$$

$$+\frac{3}{2}\mathbf{A}\lambda^2\rho\sigma_\eta \exp\left(\frac{(4\alpha+1)\,\omega}{2\,(1-\psi)} + \frac{\left(8\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) + 3\boldsymbol{B}\lambda\exp\left(\frac{(\alpha+1)\,\omega}{(1-\psi)} + \frac{\left(2\alpha^2+1\right)\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$

$$+\frac{3}{2}\boldsymbol{C}\rho\sigma_\eta \exp\left(\frac{3\omega}{2\,(1-\psi)} + \frac{3\sigma_\eta^2}{8\left(1-\psi^2\right)}\right),$$

where

$$\mathbf{A} = (4\alpha+1)\exp\left(\tfrac{\alpha(\alpha+1)\sigma_\eta^2}{(1-\psi^2)}\right) - 2\,(2\alpha+1)\exp\left(\tfrac{\alpha\sigma_\eta^2}{2(1-\psi^2)}\right) - \exp\left(\tfrac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) + 2$$

$$\boldsymbol{B} = \left[\left(1+\left(\rho\sigma_\eta\,(\alpha+1)\right)^2\right)\exp\left(\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right) - \left(1+\left(\rho\sigma_\eta\right)^2\right)\right]\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$

$$\textit{and}$$

$$\boldsymbol{C} = \left(3+\left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{3\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - \left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) + \frac{1}{6}\left(\rho\sigma_\eta\right)^2.$$

Now for $\lambda = 0$, as in JPR, the skewness coefficient of y_t, $sk\left(y_t\right)$, is, of course, the same as this one of ε_t^*, see equation 12, above, whereas for $\rho = 0$, as in Koopman and Uspensky [26], we get that $sk\left(y_t\right)$ for the SV2-PM model is the same as for the SV1-PM one, and it can be only positive (see equation 6).

Now, employing the same way, i.e. expanding $E\left(y_t - E\left(y_t\right)\right)^4$and using the appropriate expressions for $E\left(\sigma_t^s\right)$ and $E\left(\varepsilon_t^k\sigma_t^l\right)$ (see Demos [11]), we get

$$E\left(y_t - E\left(y_t\right)\right)^4 = \boldsymbol{A}\lambda^4\exp\left(\frac{4\alpha\omega}{(1-\psi)} + \frac{2\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)$$

$$+2\boldsymbol{B}\lambda^3\rho\sigma_\eta\exp\left(\frac{(6\alpha+1)\,\omega}{2\,(1-\psi)} + \frac{\left(12\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

$$+3\boldsymbol{C}\lambda^2\exp\left(\frac{(2\alpha+1)\,\omega}{(1-\psi)} + \frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) + 2\boldsymbol{D}\lambda\rho\sigma_\eta\exp\left(\frac{(2\alpha+3)\,\omega}{2\,(1-\psi)} + \frac{\left(4\alpha^2+3\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

$$+\left[\mathbf{F} - 3\mathbf{G}\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)\right]\exp\left(\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)$$

where

$$\boldsymbol{A} = \exp\left(\tfrac{6\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) - 4\exp\left(\tfrac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) + 6\exp\left(\tfrac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) - 3,$$

$$\boldsymbol{B} = (6\alpha+1)\exp\left(\frac{3\alpha(2\alpha+1)\sigma_\eta^2}{2(1-\psi^2)}\right) - 3(4\alpha+1)\exp\left(\frac{\alpha(\alpha+1)\sigma_\eta^2}{(1-\psi^2)}\right) - \exp\left(\frac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)$$
$$+3(2\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)}\right) + 3\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) - 3,$$

$$\boldsymbol{C} = 2\left(1+(\rho\sigma_\eta(2\alpha+1))^2\right)\exp\left(\frac{[4\alpha(\alpha+2)+1]\sigma_\eta^2}{4(1-\psi^2)}\right) + 2\left[1+(\rho\sigma_\eta)^2\right]\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)$$
$$-4\left(1+(\rho\sigma_\eta(\alpha+1))^2\right)\exp\left(\frac{(4\alpha+1)\sigma_\eta^2}{4(1-\psi^2)}\right) + 2(\rho\sigma_\eta)^2(2\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)}\right)$$
$$-(\rho\sigma_\eta)^2(4\alpha+1)\exp\left(\frac{\alpha(\alpha+1)\sigma_\eta^2}{(1-\psi^2)}\right) + \frac{1}{2}(\rho\sigma_\eta)^2\left[\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) - 3\right],$$

$$\mathbf{F} = \left[3 + 24(\rho\sigma_\eta)^2 + 16(\rho\sigma_\eta)^4\right]\exp\left(\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right) - \frac{3}{16}(\rho\sigma_\eta)^4,$$

$$\mathbf{G} = (\rho\sigma_\eta)^2\left\{\left(3+\left(\rho\sigma_\eta\tfrac{3}{2}\right)^2\right)\exp\left(\tfrac{\sigma_\eta^2}{2(1-\psi^2)}\right) - \tfrac{1}{2}\left(1+(\rho\sigma_\eta)^2\right)\right\}$$

and

$$\boldsymbol{D} = (2\alpha+3)\left(3+\left(\rho\sigma_\eta\frac{(2\alpha+3)}{2}\right)^2\right)\exp\left(\frac{3(2\alpha+1)\sigma_\eta^2}{4(1-\psi^2)}\right)$$
$$-3\left(1+(\rho\sigma_\eta(\alpha+1))^2\right)\exp\left(\frac{(4\alpha+1)\sigma_\eta^2}{4(1-\psi^2)}\right)$$
$$+3\left\{\left(1+(\rho\sigma_\eta)^2\right) - \left(3+\left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)\right\}\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)$$
$$+\frac{3}{4}(\rho\sigma_\eta)^2\left\{(2\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)}\right) - 1\right\},$$

For $\lambda = 0$, $\kappa(y_t)$ is the same as the kurtosis coefficient of ε_t^*, above, whereas for $\rho = 0$ we have that $\kappa(y_t)$ is the same as the one of SV1-PM model (see equation 7).

2.2.2. Dynamic Moments

The dynamic moments for the SV2-PM specification are more complicated than the SV1-PM one. This due to the fact that for the former model we have that $Cov(g(\varepsilon_t\sigma_t), f(\varepsilon_{t-k}, \sigma_{t-k})) \neq 0$, for various functions $g(.)$ and $f(.)$.

Now, to get the autocovariances the observed process, y_t, expand $Cov(y_t, y_{t-k})$ and employ equations 14, 19, 18 and 20 (see Appendix B) to get

$$Cov\left(y_t, y_{t-k}\right) = \mathbf{A}\lambda \exp\left(\frac{2\alpha\omega}{2\left(1-\psi\right)} + \frac{\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)$$
$$+\frac{1}{4}\left(\rho\sigma_\eta\right)^2\left\{\left(\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{\omega}{\left(1-\psi\right)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right).$$

where

$$\mathbf{A} = \lambda\left\{\exp\left(\frac{\alpha^2\psi^k\sigma_\eta^2}{\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{\alpha\omega}{\left(1-\psi\right)}+\frac{\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)$$
$$+\rho\sigma_\eta\left\{\left(\alpha\psi^k+1\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{\omega}{2\left(1-\psi\right)}+\frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right).$$

It is possible that, depending on the parameter values, the k^{th} order autocovariance can be either positive or negative. Further

$$Var\left(y_t\right) = \lambda^2\left[\exp\left(\frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right)-1\right]\exp\left(\frac{2\alpha\omega}{\left(1-\psi\right)}+\frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right)$$
$$+\lambda\rho\sigma_\eta\left[\left(2\alpha+1\right)\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right]\exp\left(\frac{\left(2\alpha+1\right)\omega}{2\left(1-\psi\right)}+\frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$
$$+\left\{\left(1+\rho^2\sigma_\eta^2\right)\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-\frac{\left(\rho\sigma_\eta\right)^2}{4}\right\}\exp\left(\frac{\omega}{\left(1-\psi\right)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$

Now in the model of Koopman and Uspensky [26] we have $\rho = 0$ and it follows that that the autocorrelation function of the SV2-PM model is the same as for the SV1-PM one and consequently, $Corr\left(y_t, y_{t-k}\right)$ can be only positive. The same is true in the case of JPR, where we have that $\lambda = 0$, i.e.

$$Corr\left(y_t, y_{t-k}\right) = \frac{1}{4}\left(\rho\sigma_\eta\right)^2\frac{\left(\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{4(1-\psi^2)}\right)-1}{\left(1+\rho^2\sigma_\eta^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)-\frac{(\rho\sigma_\eta)^2}{4}},$$

and $Corr\left(y_t, y_{t-k}\right)$ can be only positive. However, in this case, i.e. if $\lambda = 0$, the autocorrelations of the observed process for the SV1-PM model is zero.

For the leverage effect we get

$$Corr\left(\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right) = \frac{1}{2}\rho\sigma_\eta\frac{\left(2\psi^k+1\right)\exp\left[\frac{\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right]-1}{\sqrt{\left[\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\left[\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)-\left(\frac{1}{2}\rho\sigma_\eta\right)^2\right]}},$$

which is negative provided that $\rho < 0$, something which also is true for the SV1-PM model (see equation 9).

Employing, again, equations 14, 19, 18 and 20 (see Appendix B) we get the dynamic asymmetry for the SV2-PM, i.e.

$$
\begin{aligned}
Cov\left(y_t^2, y_{t-k}\right) = \mathbf{D}\lambda^2 \exp\left(\frac{2\alpha\omega}{1-\psi} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) \qquad (13) \\
+2c\lambda\rho\sigma_\eta \left\{\left(\alpha\psi^k + 1\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right) - 1\right\}\exp\left(\frac{(2\alpha+1)\,\omega}{2(1-\psi)} + \frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{8(1-\psi^2)}\right) \\
+\boldsymbol{F}\lambda^2\rho\sigma_\eta \exp\left(\frac{(4\alpha+1)\,\omega}{2(1-\psi)} + \frac{\left(8\alpha^2+1\right)\sigma_\eta^2}{8(1-\psi^2)}\right) \\
+\frac{\rho\sigma_\eta}{2}\left(1+(\rho\sigma_\eta)^2\right)\left[\left(2\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right) - 1\right]\exp\left(\frac{3\omega}{2(1-\psi)} + \frac{5\sigma_\eta^2}{8(1-\psi^2)}\right) \\
+c\frac{(\rho\sigma_\eta)^2}{2}\left[\left(\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{4(1-\psi^2)}\right) - 1\right]\exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4(1-\psi^2)}\right) \\
+\boldsymbol{G}\lambda \exp\left(\frac{(\alpha+1)\,\omega}{1-\psi} + \frac{\left(2\alpha^2+1\right)\sigma_\eta^2}{4(1-\psi^2)}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{D} = \lambda\left[\exp\left(\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right]\exp\left(\frac{\alpha\omega}{1-\psi} + \frac{3\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right) + 2c\left[\exp\left(\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right], \\
\boldsymbol{F} = (2\alpha+1)\left[\exp\left(\frac{\alpha\,(2\alpha+1)\,\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right) - 1\right]\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)}\right) \\
+\frac{1}{2}\left[\left(4\alpha\psi^k+1\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right]\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{G} = (\rho\sigma_\eta)^2\,\frac{(2\alpha+1)}{2}\left\{\left[(2\alpha+1)\,\psi^k+1\right]\exp\left(\frac{(2\alpha+1)\,\psi^k\sigma_\eta^2}{4(1-\psi^2)}\right) - 1\right\}\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)}\right) \\
+\left(1+(\rho\sigma_\eta)^2\right)\left[\exp\left(\frac{2\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right]\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right).
\end{aligned}
$$

Now for the Koopman and Uspensky [26] model, i.e. for $\rho = 0$ and $c \neq 0$, we get that $Cov\left(y_t^2, y_{t-k}\right)$ for the model is the same as for the SV1-PM model, whereas for $\lambda = 0$ but $\rho \neq 0$ and $c \neq 0$ we get that $Cov\left(y_t^2, y_{t-k}\right)$ can be of either sign, depending on the parameter values.

As the stochastic volatility process is the same for the two model, the volatility clustering is given by equation 10 for the SV2-M model, as well. However, now

$$Corr\left(\varepsilon_t^{*2}, \varepsilon_{t-k}^{*2}\right) = \left(1+\rho^2\sigma_\eta^2\right) \frac{\left(1+\rho^2\sigma_\eta^2\left(\psi^k+1\right)^2\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - \left(1+\rho^2\sigma_\eta^2\right)}{\left[3+24\left(\rho\sigma_\eta\right)^2+16\left(\rho\sigma_\eta\right)^4\right]\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right) - \left(1+\rho^2\sigma_\eta^2\right)^2}$$

which is different for the SV1-PM (see equation 11). It seems that the autocorrelations of the $\varepsilon_t^{*2\prime}s$ for the SV1-PM model is higher for the ones of the SV2-PM, at least for plausible parameter values and small values of k. However, the decay of the SV2-PM autocorrelations is smaller.

For the dynamic kurtosis, employ, again, equations 14, 19, 18 and 20 (see Appendix B) to get

$$\begin{aligned}
Cov\left(y_t^2, y_{t-k}^2\right) &= \lambda^4\left[\exp\left(\frac{4\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\exp\left(\frac{4\alpha\omega}{1-\psi}+\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)\\
&+2\mathcal{A}\lambda^3\exp\left(\frac{3\alpha\omega}{1-\psi}+\frac{20\alpha^2\sigma_\eta^2}{8(1-\psi^2)}\right)+\mathcal{B}\lambda^2\exp\left(\frac{2\alpha\omega}{1-\psi}+\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)\\
&+\mathcal{C}\lambda^2\rho\sigma_\eta\exp\left(\frac{(4\alpha+1)\omega}{2(1-\psi)}+\frac{\left(8\alpha^2+1\right)\sigma_\eta^2}{8(1-\psi^2)}\right)+\mathcal{D}\lambda\rho\sigma_\eta\exp\left(\frac{(2\alpha+3)\omega}{2(1-\psi)}+\frac{\left((2\alpha+1)^2+4\right)\sigma_\eta^2}{8(1-\psi^2)}\right)\\
&+\mathcal{F}c\lambda\rho\sigma_\eta\exp\left(\frac{(2\alpha+1)\omega}{2(1-\psi)}+\frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{8(1-\psi^2)}\right)+2\mathcal{G}c\rho\sigma_\eta\exp\left(\frac{3\omega}{2(1-\psi)}+\frac{5\sigma_\eta^2}{8(1-\psi^2)}\right)\\
&+2\mathcal{H}c\lambda\exp\left(\frac{(\alpha+1)\omega}{1-\psi}+\frac{\left(\alpha^2+1\right)\sigma_\eta^2}{2(1-\psi^2)}\right)\\
&+c^2\left(\rho\sigma_\eta\right)^2\left[\left(\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{4(1-\psi^2)}\right)-1\right]\exp\left(\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)\\
&+\left(1+\left(\rho\sigma_\eta\right)^2\right)\left\{\begin{array}{c}\left(1+\left(\rho\sigma_\eta\right)^2\left(\psi^k+1\right)^2\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)\\ -\left(1+\left(\rho\sigma_\eta\right)^2\right)\end{array}\right\}\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A} &= \rho\sigma_\eta\left[\begin{array}{c}\left[2\alpha\psi^k+(2\alpha+1)\right]\exp\left(\frac{\alpha(2\alpha+1)\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)\\ -(2\alpha+1)\end{array}\right]\exp\left(\frac{\omega}{2(1-\psi)}+\frac{(4\alpha+1)\sigma_\eta^2}{8(1-\psi^2)}\right)\\
&+2c\left[\exp\left(\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)-1\right],\\
\mathcal{B} &= \left[\begin{array}{c}\left[\left(\rho\sigma_\eta\right)^2\left(\left(2\alpha\psi^k+1\right)^2+1\right)+2\right]\exp\left(\frac{2\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)\\ -2\left(1+\left(\rho\sigma_\eta\right)^2\right)\end{array}\right]\exp\left(\frac{\omega}{1-\psi}+\frac{\left(2\alpha^2+1\right)\sigma_\eta^2}{2(1-\psi^2)}\right)\\
&+4c^2\left[\exp\left(\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)-1\right],
\end{aligned}$$

and

$$\mathcal{C} = 2c\left[\left(2\alpha\psi^k+1\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)$$

$$+\rho\sigma_\eta\left(2\alpha+1\right)^2\left[\left(\psi^k+1\right)\exp\left(\frac{\left(2\alpha+1\right)^2\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right]\exp\left(\frac{\omega}{2\left(1-\psi\right)}+\frac{\left(8\alpha+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

$$+4c\left\{\left(\alpha\psi^k+2\alpha+1\right)\exp\left(\frac{\alpha\left(2\alpha+1\right)\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-\left(2\alpha+1\right)\right\}\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)$$

$$\mathcal{D} = \left\{\begin{array}{c}\left(2\alpha+1\right)\left[1+\left(\rho\sigma_\eta\right)^2\frac{\left(\left(2\alpha+1\right)\psi^k+2\right)^2}{4}\right]\\+\left[1+\left(\rho\sigma_\eta\right)^2\right]\left(2\psi^k+2\alpha+1\right)\end{array}\right\}\exp\left(\frac{\left(2\alpha+1\right)\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-2\left(1+\left(\rho\sigma_\eta\right)^2\right)\left(2\alpha+1\right),$$

$$\mathcal{F} = \rho\sigma_\eta\left\{\begin{array}{c}\left[\begin{array}{c}\left(\left(2\alpha+1\right)^2+1\right)\psi^k\\+2\left(2\alpha+1\right)\end{array}\right]\exp\left(\frac{\left(2\alpha+1\right)\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)\\-2\left(2\alpha+1\right)\end{array}\right\}\exp\left(\frac{\omega}{2\left(1-\psi\right)}+\frac{\left(4\alpha+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

$$+4c\left[\left(\alpha\psi^k+1\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right],$$

$$\mathcal{G} = \left[\left(1+\left(\rho\sigma_\eta\right)^2\right)\frac{\left(2\psi^k+1\right)}{2}+\frac{1}{2}\left(1+\left(\rho\sigma_\eta\right)^2\frac{\left(\psi^k+2\right)^2}{4}\right)\right]\exp\left(\frac{\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-\left(1+\left(\rho\sigma_\eta\right)^2\right),$$

$$\mathcal{H} = \left(\left(\rho\sigma_\eta\right)^2\left(1+\left(\alpha\psi^k+1\right)^2\right)+2\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)-2\left(1+\left(\rho\sigma_\eta\right)^2\right).$$

Also

$$Var\left(y_t^2\right) = \mathbf{A}\lambda^3\exp\left(\frac{3\alpha\omega}{1-\psi}+\frac{5\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)+2\mathbf{B}c\exp\left(\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$

$$+2\mathbf{C}\lambda^2\rho\sigma_\eta\exp\left(\frac{\left(4\alpha+1\right)\omega}{2\left(1-\psi\right)}+\frac{\left(8\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)+4\mathbf{D}c^2\lambda\exp\left(\frac{\alpha\omega}{1-\psi}+\frac{\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)$$

$$+2\mathbf{F}\lambda\exp\left(\frac{\left(\alpha+1\right)\omega}{(1-\psi)}+\frac{\left(2\alpha^2+1\right)\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)+2\mathbf{G}\lambda^2\exp\left(\frac{\left(2\alpha+1\right)\omega}{1-\psi}+\frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$

$$+\left\{\left[3+24\left(\rho\sigma_\eta\right)^2+16\left(\rho\sigma_\eta\right)^4\right]\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-\left(1+\left(\rho\sigma_\eta\right)^2\right)^2\right\}\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right).$$

where

$$\mathbf{A} = \lambda \left[\exp\left(\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)} \right) - 1 \right] \exp\left(\frac{\alpha\omega}{1-\psi} + \frac{3\alpha^2\sigma_\eta^2}{2(1-\psi^2)} \right) + 4c \left[\exp\left(\frac{2\alpha^2\sigma_\eta^2}{(1-\psi^2)} \right) - 1 \right],$$

$$\mathbf{B} = \rho\sigma_\eta \left[3\left(3+\frac{9}{4}(\rho\sigma_\eta)^2\right) \exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)} \right) - \left(1+(\rho\sigma_\eta)^2\right) \right] \exp\left(\frac{\omega}{2(1-\psi)} + \frac{3\sigma_\eta^2}{8(1-\psi^2)} \right)$$
$$+2c\left[\left(1+(\rho\sigma_\eta)^2\right) \exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)} \right) - \frac{(\rho\sigma_\eta)^2}{4} \right],$$

$$\mathbf{C} = \lambda \left\{ (6\alpha+1) \exp\left(\frac{\alpha(2\alpha+1)\sigma_\eta^2}{(1-\psi^2)} \right) - (2\alpha+1) \right\} \exp\left(\frac{\alpha\omega}{(1-\psi)} + \frac{\alpha(3\alpha+1)\sigma_\eta^2}{2(1-\psi^2)} \right)$$
$$+c\left[3(4\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{(1-\psi^2)} \right) - 1 \right] \exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)} \right) - 2c(2\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)} \right),$$

$$\mathbf{D} = \lambda \left[\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)} \right) - 1 \right] \exp\left(\frac{\alpha\omega}{1-\psi} + \frac{\alpha^2\sigma_\eta^2}{2(1-\psi^2)} \right)$$
$$+\rho\sigma_\eta \left[(2\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)} \right) - 1 \right] \exp\left(\frac{\omega}{2(1-\psi)} + \frac{\sigma_\eta^2}{8(1-\psi^2)} \right),$$

$$\mathbf{G} = \left[3\left(1+(\rho\sigma_\eta)^2(2\alpha+1)^2\right) \exp\left(\frac{2\alpha\sigma_\eta^2}{(1-\psi^2)} \right) - \left(1+(\rho\sigma_\eta)^2\right) \right] \exp\left(\frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{4(1-\psi^2)} \right)$$
$$-(\rho\sigma_\eta)^2 \frac{(2\alpha+1)^2}{2} \exp\left(\frac{\alpha\sigma_\eta^2}{(1-\psi^2)} \right),$$

and

$$\mathbf{F} = \mathbf{F_1}\rho\sigma_\eta \exp\left(\frac{\omega}{2(1-\psi)} + \frac{(4\alpha+3)\sigma_\eta^2}{8(1-\psi^2)} \right) + \mathbf{F_2} c \exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)} \right)$$
$$-2c\left(1+(\rho\sigma_\eta)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)} \right),$$

$$\mathbf{F_1} = (2\alpha+3)\left[3+(\rho\sigma_\eta)^2\frac{(2\alpha+3)^2}{4} \right] \exp\left(\frac{(2\alpha+1)\sigma_\eta^2}{2(1-\psi^2)} \right) - \left(1+(\rho\sigma_\eta)^2\right)(2\alpha+1),$$

$$\mathbf{F_2} = 6\left(1+(\rho\sigma_\eta(\alpha+1))^2\right)\exp\left(\frac{(2\alpha+1)\sigma_\eta^2}{4(1-\psi^2)} \right) - (\rho\sigma_\eta)^2(2\alpha+1).$$

Further if $\lambda = c = 0$, as in JPR, we get

$$Corr\left(y_t^2, y_{t-k}^2\right) = \frac{\left(1+(\rho\sigma_\eta)^2\right)\left\{ \left(1+\left(\left(\psi^k+1\right)\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - \left(1+(\rho\sigma_\eta)^2\right) \right\}}{\left[3+24(\rho\sigma_\eta)^2+16(\rho\sigma_\eta)^4\right]\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right) - \left(1+(\rho\sigma_\eta)^2\right)^2},$$

which is positive for any k.

For $c \neq 0$ and $\rho = 0$, as in Koopman and Uspensky [26], the $Cov\left(y_t^2, y_{t-k}^2\right)$ and $V\left(y_t^2\right)$ are the same as for the SV1 process and are presented in Appendix A.. The same applies for the case of $c = \rho = 0$.

3. Comparisons, Empirical Results and Conclusions

The SV2-PM model exhibits more complex static and dynamic moments than SV1-PM, due to the contemporaneous correlation between the in-mean error and conditional variance when $\rho \neq 0$. If $\rho = 0$, both models yield identical moments and are observationally equivalent.

Assuming $\lambda > 0$ (positive price of risk), $\rho < 0$ (leverage effect), and $c = 0$ (market efficiency), the expected return $E\left(y_t\right)$ (risk premium) is positive in SV1-PM but can be either sign in SV2-PM.

When $\lambda = 0$, the skewness is zero in SV1-PM but negative in SV2-PM. More generally, with $\lambda \neq 0$ and $\rho \neq 0$, return skewness can vary in sign, though the mean error skewness remains negative. For instance, JBR (Table 4) estimates return skewness at –0.54 using a fat-tailed error distribution, giving SV2-PM an edge by replicating negative skewness without asymmetric error terms (see Harvey and Palumbo 2023 [20]; Blasques et al 2023 [6]).

For kurtosis, SV1-PM matches the mean error's kurtosis when $\lambda = 0$, whereas SV2-PM shows lower kurtosis in the observed process. Autocorrelations, $Corr\left(y_t, y_{t-k}\right)$, vary with λ and ρ; they vanish in SV1-PM if $\lambda = 0$, but remain positive in SV2-PM.

The leverage effect, $Corr\left(\sigma_t^2, \varepsilon_{t-k}^*\right)$, is negative in both models and disappears if $\rho = 0$. Dynamic asymmetries depend on the interaction of ρ and λ, but are negative in both models when $\lambda = 0$ and $\rho \neq 0$. Both models share identical volatility clustering expressions, $Corr\left(\sigma_t^2, \sigma_{t-k}^2\right)$.

Table 1 shows SV1-PM moment estimates. Columns 1–3 use restricted parameters ($c = \lambda = 0$) from Yu [40] (Table 1) and Asai and McAleer [2] (Table 2), yielding zero skewness and autocorrelations. Columns 4–5 (Arvanitis and Demos [1]) use unrestricted estimates, producing positive skewness and small autocorrelations.

Table 1. Estimated Moments for SV1-PM Models

	Yu	AM		AD	
	S&P	S&P	Topix	S&P	DAX
$sk\left(y_t\right)$	0	0	0	0.027	0.037
$\kappa\left(y_t\right)$	4.497	3.536	4.025	5.178	5.478
$Corr\left(y_t, y_{t-1}\right)$	0	0	0	-0.004	-0.003
$Corr\left(y_t, y_{t-2}\right)$	0	0	0	-0.003	-0.003
$Corr\left(\sigma_t^2, \varepsilon_{t-1}^*\right)$	-0.078	-0.154	-0.176	-0.213	-0.140
$Corr\left(\sigma_t^2, \varepsilon_{t-2}^*\right)$	-0.075	-0.152	-0.166	-0.197	-0.139
$Corr\left(y_t^2, y_{t-1}\right)$	-0.029	-0.060	-0.059	-0.082	-0.050
$Corr\left(y_t^2, y_{t-1}\right)$	-0.002	-0.040	-0.055	-0.075	-0.046
$Corr\left(y_t^2, y_{t-1}^2\right)$	0.139	0.071	0.110	0.163	0.169
$Corr\left(y_t^2, y_{t-2}^2\right)$	0.134	0.070	0.104	0.150	0.159

Table 2 provides SV2-PM estimates. In Columns 1–2 (Koopman & Uspensky [26], whith $\rho = 0$), leverage is absent, and autocorrelations are small but positive. Columns 3–4 (Yu [40]; and Jacquier et al [24]) apply restricted estimates with $c = \lambda = 0$, showing negative skewness and leverage, consistent with theoretical expectations.

Table 2. Estimated Moments for SV2-PM Models				
	KU		Yu	JPR
	FT	S&P	S&P	VW
$sk(y_t)$	-0.0234	-0.040	-0.131	-0.502
$\kappa(y_t)$	5.425	4.975	4.487	5.789
$Corr(y_t, y_{t-1})$	0.000	0.000	0.000	0.004
$Corr(y_t, y_{t-2})$	0.000	0.000	0.000	0.003
$Corr\left(\sigma_t^2, \varepsilon_{t-1}^*\right)$	0	0	-0.061	-0.172
$Corr\left(\sigma_t^2, \varepsilon_{t-2}^*\right)$	0	0	-0.059	-0.160
$Corr\left(y_t^2, y_{t-1}\right)$	-0.003	-0.007	-0.023	-0.063
$Corr\left(y_t^2, y_{t-2}\right)$	-0.001	-0.006	-0.011	-0.058
$Corr\left(y_t^2, y_{t-1}^2\right)$	0.173	0.158	0.148	0.166
$Corr\left(y_t^2, y_{t-2}^2\right)$	0.169	0.153	0.138	0.153

In total, both models can replicate core return features, but SV2-PM offers greater flexibility, especially in capturing negative skewness without requiring asymmetric error distributions. A promising extension for both frameworks is introducing autocorrelation in the standardized mean error (e.g., via ARMA), which could improve model fit at the cost of added complexity.

Appendix A

Employing formulae 6-14 in Demos [10], with the appropriate substitutions (please see Demos [11]), we get:

$$E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\right) = \exp\left[\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(2sd\psi^k + d^2 + s^2\right)}{2\left(1-\psi^2\right)}\right], \tag{14}$$

and

$$\begin{aligned} E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\varepsilon_{t-k}\right) &= s\rho\phi_\eta\psi^{k-1}E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\right) \quad \textit{and} \\ E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\varepsilon_{t-k}^2\right) &= \left[1 + s^2\rho^2\phi_\eta^2\psi^{2k-2}\right]E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\right). \end{aligned} \tag{15}$$

Appendix B

First, form Perez, Ruiz and Veiga [33] we have that

$$E\left[\varepsilon_t \exp\left(A\eta_t\right)\right] = \rho\sigma_\eta A \exp\left[\frac{A^2\sigma_\eta^2}{2}\right], \tag{16}$$

and as $\Phi\left(\frac{3}{2},\frac{1}{2};z\right)=2\left(z+\frac{1}{2}\right)\Phi\left(\frac{1}{2},\frac{1}{2};z\right) = 2\left(z+\frac{1}{2}\right)\exp\left(z\right)$ see Gradshteyn and Ryzhik (1994) [18], formula 9.212.4, where $\Phi\left(.,.;z\right)$ is the confluent hypergeometric function, and from Perez, Ruiz and Veiga [33], equation 19, we get

$$E\left(\varepsilon_t^2\exp\left(A\eta_t\right)\right) = \left(1+\left(\rho\sigma_\eta A\right)^2\right)\exp\left[\frac{A^2\sigma_\eta^2}{2}\right].$$

Additionally,

$$\begin{aligned} E\left(\varepsilon_t^3\exp\left(A\eta_t\right)\right) &= E\left(\varepsilon_t^3 E\left[\exp\left(A\eta_t\right)|\varepsilon_t\right]\right) = \exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right]E\left(\varepsilon_t^3\exp\left[\rho\sigma_\eta A\varepsilon_t\right]\right) \\ &= \left(\rho\sigma_\eta A\right)\left(3+\left(\rho\sigma_\eta A\right)^2\right)\exp\left[\frac{A^2\sigma_\eta^2}{2}\right]. \end{aligned}$$

Notice that $E\left(\sigma_t\right)$, $E\left(\sigma_t^2\right)$, $E\left(\sigma_t^3\right)$ and $E\left(\sigma_t^4\right)$ are the same as for the SV1-PM model.

First,

$$\begin{aligned} E\left(\sigma_t^B\varepsilon_t^A\right) &= E\left(\exp\left(\frac{B\omega}{2} + \frac{B}{2}\psi\ln\sigma_{t-1}^2 + \frac{B}{2}\eta_t\right)\varepsilon_t^A\right) \\ &= E\left(\exp\left(\frac{B}{2}\eta_t\right)\varepsilon_t^A\right)\exp\left(\frac{B\omega}{2}\frac{1}{(1-\psi)} + \frac{\left(B\psi\right)^2\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \end{aligned} \tag{17}$$

and employing equation 16 we get

$$E(\sigma_t \varepsilon_t) = \frac{\rho \sigma_\eta}{2} \exp\left(\frac{1}{2}\frac{\omega}{2(1-\psi)} + \frac{\sigma_\eta^2}{8(1-\psi^2)}\right).$$

From formula 9.212.4 in Gradshteyn and Ryzhik [18] we get

$$\Phi\left(\frac{5}{2}, \frac{1}{2}, \frac{A^2\rho^2\sigma_\eta^2}{2}\right) = \frac{2}{3}\left[\left(\frac{A^2\rho^2\sigma_\eta^2}{2} + \frac{5}{2}\right)\Phi\left(\frac{3}{2}, \frac{1}{2}, \frac{A^2\rho^2\sigma_\eta^2}{2}\right) - \Phi\left(\frac{1}{2}, \frac{1}{2}, \frac{A^2\rho^2\sigma_\eta^2}{2}\right)\right],$$

and from Perez, Ruiz and Veiga [33], equation 19, we get

$$E\left(\varepsilon_t^4 \exp(A\eta_t)\right) = \exp\left[\frac{A^2\sigma_\eta^2}{2}\right]\left[3 + 6(\rho\sigma_\eta A)^2 + (\rho\sigma_\eta A)^4\right].$$

Finally, notice that

$$E\left(\varepsilon_t^A \sigma_t^B \sigma_{t-k}^D\right) = E\left(\varepsilon_t^A \exp\left(\frac{B}{2}\eta_t\right)\right) \tag{18}$$
$$\times \exp\left((B+D)\frac{\omega}{2(1-\psi)} + \frac{\sigma_\eta^2}{8(1-\psi^2)}\left(2DB\psi^k + D^2 + (B\psi)^2\right)\right)$$

$$E\left(\sigma_t^B \varepsilon_{t-k}^C \sigma_{t-k}^D\right) = E\left(\exp\left(\frac{B\psi^k + D}{2}\eta_{t-k}\right)\varepsilon_{t-k}^C\right)$$
$$\times \exp\left((B+D)\frac{\omega}{2(1-\psi)} + \left(B^2 + \left(\psi^2 - 1\right)B^2\psi^{2k} + 2BD\psi^{k+2} + D^2\psi^2\right)\frac{\sigma_\eta^2}{8(1-\psi^2)}\right) \tag{19}$$

$$E\left(\varepsilon_t^A \sigma_t^B \varepsilon_{t-k}^C \sigma_{t-k}^D\right) = E\left(\varepsilon_t^A \exp\left(\frac{B}{2}\eta_t\right)\right) E\left(\exp\left(\frac{B\psi^k + D}{2}\eta_{t-k}\right)\varepsilon_{t-k}^C\right)$$
$$\times \exp\left((B+D)\frac{\omega}{2(1-\psi)} + \left((B\psi)^2 + \left(\psi^2 - 1\right)B^2\psi^{2k} + 2BD\psi^{k+2} + D^2\psi^2\right)\frac{\sigma_\eta^2}{8(1-\psi^2)}\right) \tag{20}$$

References

[1] Arvanitis, S. and A. Demos (2024). Gaussian Stochastic Volatility, Misspecified Volatility Filters and Indirect Inference Estimation, WP 17-2024, Dept. of Economics, AUEB.

[2] Asai, M. and M. McAleer (2011). Alternative Asymmetric Stochastic Volatility Models, *Econometric Reviews* 30, 548-564.

[3] Barndorff-Nielsen, O. E. (1997). Normal Inverse Gaussian Distributions and Stochastic Volatility Modelling, *The Scandinavian Journal of Statistics* 24, 1-13.

[4] Bermudez, P., J.M. Marín and H. Veiga (2020). Data cloning estimation for asymmetric stochastic volatility models. *Econometric Reviews*, DOI: 10.1080/07474938.2020.1770997

[5] Black, F. (1976). Studies of stock, price volatility changes, *Proceedings of the 1976 Meetings of the Business and Economics Statistics Section*, American Statistical Association, 177-181.

[6] Blasques, F., C. Francq and S. Laurent (2023). Quasi score-driven models, *Journal of Econometrics* 234, 251-275.

[7] Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics* 31, 307-327.

[8] Campbell, J.Y. and L. Hentschel (1992). No news is good News, an asymmetric model of changing volatility stock returns, *Journal of Financial Economics* 31, 281-318

[9] Carnero, M.A., D. Pena and E. Ruiz (2004). Persistence and kurtosis in GARCH and Stochastic Volatility Models, *Journal of Financial Econometrics* 2, 319-342.

[10] Demos, A. (2002). Moments and Dynamic Structure of a Time-Varying-Parameter Stochastic Volatility in Mean Model, *The Econometrics Journal* 5.2, 345-357.

[11] Demos, A. (2025). Statistical Properties of Two Asymmetric Stochastic Volatility in Power Mean Models, Discussion Paper 25-46, Dept. of International and European Economic Studies, AUEB. https://wpa.deos.aueb.gr/docs/2025.SV.GT.G.Sta.4.pdf

[12] Engle, R. F. (1982). Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation, *Econometrica* 50, 987-1007.

[13] French K.R., G.W. Schwert and R.F. Stambaugh (1987). Expected Stock Returns and Volatility, *Journal of Financial Economics* 19, 3-29.

[14] Gallant, R. A. and G. Tauchen (1996). Which Moments to Match, *Econometric Theory* 12, 657-81.

[15] Ghysels, E., A. Harvey, and E. Renault (1996). Stochastic Volatility, in G. Maddala and C. Rao (Eds.), *Handbook of Statistics* 14, 119–191, Elsevier Science.

[16] Glosten, L. R., R. Jagannathan and D. E. Runkle (1993). On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, *Journal of Finance* 48, 1779-1801.

[17] Gourieroux, C., A. Monfort, and E. Renault (1993). Indirect Inference, *Journal of Applied Econometrics* 8, S85-S118.

[18] Gradshteyn, I. and I. Ryzhik (1994). Table of Integrals, Series and Products. Academic Press.

[19] Harvey, A. (2007). Long Memory in Stochastic Volatility, in J. Knight and S. Satchell (Eds.), *Forecasting Volatility in the Financial Markets*, Third Edition, 351-363, Elsevier.

[20] Harvey, A. and Palumbo (2023). Score-driven models for realized volatility, *Journal of Econometrics* 237, art. 105448.

[21] Harvey, A., E. Ruiz and N. Shephard (1994). Multivariate Stochastic Variance Models, *Review of Economic Studies* 61, 247-264.

[22] Harvey, A. and N. Shephard (1996). Estimation of an Asymmetric Stochastic Volatility Model for asset Returns, *Journal of Business and Economic Statistics* 14, 429-434.

[23] Jacquier, E., N.G. Polson and P.E. Rossi (1994). Bayesian Analysis of Stochastic Volatility Models, *Journal of Business and Economic Statistics* 12, 371-417.

[24] Jacquier, E., N.G. Polson and P.E. Rossi (2004). Bayesian Analysis of Stochastic Volatility Models with Fat-tails and Correlated Errors, Journal of Econometrics 122, 185-212.

[25] Kim, S., N. Shephard and S. Chib (1998). Stochastic Volatility: Likelihood Inference and

comparison with *ARCH* Models, *Review of Economic Studies* 65, 361-393.

[26] Koopman, S.J. and E. H. Uspensky (2002), The Stochastic Volatility in Mean Model: Empirical Evidence from International Stock Markets, *Journal of Applied Econometrics* 17, 667-689.

[27] Mao, X., V. Czellar, E. Ruiz, and H. Veiga (2020). Asymmetric stochastic volatility models: Properties and particle filter-based simulated maximum likelihood estimation, *Econometrics and Statistics* 13, 84–105.

[28] blackMarin, J.M., E. Romero and H. Veiga (2024). Fitting complex stochastic volatility models using Laplace approximation, WP2024-05, Statistics and Econometrics series, Universidad Carlos III de Madrid.

[29] Merton, R.C. (1980). On Estimating the Expected Return on the Market. An Exploratory Investigation, *Journal of Financial Economics* 8, 323-361.

[30] Meyer, R. and J. Yu (2000). BUGS for a Bayesian Analysis of Stochastic Volatility Models, *Econometrics Journal* 3, 198-215..

[31] Nelson, D.B. (1991). Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica* 59, 347-370.

[32] Perez, A. and E. Ruiz (2003), Properties of the Sample Autocorrelations of Nonlinear Transformations in Long Memory Stochastic Volatility Models, *Journal of Financial Econometrics* 1, 420-444.

[33] Perez, E. Ruiz and H. Veiga (2009). A note on the properties of power-transformed returns in long-memory stochastic volatility models with leverage effect, *Computational Statistics and Data Analysis* 53, 3593-3600.

[34] Romero, E. and E. Ropero-Moriones (2023). Testing Data Cloning as the Basis of an Estimator for the Stochastic Volatility in Mean Model, *Discrete Dynamics in Nature and Society*2023, Article ID 7657430, https://doi.org/10.1155/2023/7657430.

[35] Rodriguez, M.J. and E. Ruiz (2012). Revisiting Several popular *GARCH* Models with Leverage Effect: Differences and Similarities, *Journal of Financial Econometrics* 10, 637-668.

[36] Ruiz, E. and E. Veiga (2008). Modelling long memory volatilities with leverage effect: A-LMSV versus FIEGARCH, *Computational Statistics and Data Analysis* 52, 2846-2862.

[37] Sentana, E. (1995). Quadratic ARCH Models, *Review of Economic Studies* 62, 639-661.

[38] Taylor, S.J. (1986). *Modelling Financial Time Series*, J. Wiley, Chichester.

[39] Tsiotas, G. (2012). On Generalised Asymmetric Stochastic Volatility Models, *Computational Statistics and Data Analysis* 56, 151-172.

[40] Yu, J. (2005). On Leverage in a Stochastic Volatility Model, *Journal of Econometrics* 127, 165-178.